Gradient estimates for the heat equation under the Ricci flow

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Abstract

The paper considers a manifold M evolving under the Ricci flow and establishes a series of gradient estimates for positive solutions of the heat equation on M. Among other results, we prove Li-Yau-type inequalities in this context. We consider both the case where M is a complete manifold without boundary and the case where M is a compact manifold with boundary. Applications of our results include Harnack inequalities for the heat equation on M.

1 Introduction

The paper deals with a manifold M evolving under the Ricci flow and with positive solutions to the heat equation on M. We establish a series of gradient estimates for such solutions including several Li-Yau-type inequalities. First, we study the case where M is a complete manifold without boundary. Our results contain estimates of both local and global nature. Second, we look at the situation where M is compact and has nonempty boundary ∂M. We impose the condition that ∂M remain convex and umbilic at all times. Our arguments then yield two global estimates.

Suppose M is a manifold without boundary. Let $\big(M, g(x,t)\big)_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \qquad x \in M,\ t \in [0,T]. \tag{1.1}$$

We assume its curvature remains uniformly bounded for all $t \in [0,T]$. Consider a positive function $u(x,t)$ defined on $M \times [0,T]$. In Section 2, we assume $u(x,t)$ solves the equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad x \in M,\ t \in [0,T]. \tag{1.2}$$

The symbol Δ here stands for the Laplacian given by $g(x,t)$. It is important to emphasize that Δ depends on the parameter t. Thus, we look at the Ricci flow (1.1) combined with the heat equation (1.2). Note that formula (1.1) provides us with additional information about the coefficients of the operator Δ appearing in (1.2) but is itself fully independent of (1.2). To learn about the history, the intuitive meaning, the technical aspects, and the applications of the Ricci flow, one should refer to the many quality books on the subject such as, for example, [12, 33, 23, 9, 10].

Problem (1.1) combined with (1.2) admits a simple interpretation in terms of the process of heat conduction. More specifically, one may think of the manifold M with the initial metric $g(x,0)$ as an object having the temperature distribution $u(x,0)$. Suppose we let M evolve under the Ricci flow and simultaneously let the heat spread on M. Then the solution $u(x,t)$ will represent the temperature of M at the point x at time t. The work [2] provides a probabilistic interpretation of (1.1)–(1.2). In particular, it constructs a Brownian motion related to $u(x,t)$.

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The study of system (1.1)–(1.2) arose from R. Hamilton's paper [16]. The original idea in [16] was to investigate the Ricci flow combined with the heat flow of harmonic maps. The system we examine in Section 2 may be viewed as a special case. The idea to consider the Ricci flow combined with the heat flow of harmonic maps was further exploited in [29, 30] for the purposes of regularizing non-smooth Riemannian metrics. We point out, without a deeper explanation, that looking at the two evolutions together leads to interesting simplifications in the analysis.

After its conception in [16], the study of (1.1)–(1.2) was pursued in [14, 24, 36, 2, 6]. A large amount of work was done to understand several problems that are similar to (1.1)–(1.2) in one way or another. The list of relevant references includes but is not limited to [36, 5, 6] and [10, Chapter 16]. For instance, there are substantial results concerning the Ricci flow combined with the conjugate heat equation. The connection of this problem to (1.1)–(1.2) is beyond superficial. Q. Zhang used a gradient estimate for (1.1)–(1.2) to prove a Gaussian bound for the conjugate heat equation in [36]. The results of the present paper may have analogous applications.

System (1.1)–(1.2) could serve as a model for researching the Ricci flow combined with the heat flow of harmonic maps. There are other geometric evolutions for which (1.1)–(1.2) plays the same role. One example is the Ricci Yang-Mills flow; see [19, 32, 35]. The analysis of this evolution is technically complicated. Its properties are not yet well understood. We expect that investigating the simpler model case of system (1.1)–(1.2) will provide insight on the behavior of the Ricci Yang-Mills flow. We also speculate that the results of the present paper may aid in proving relevant existence theorems; cf. [1, 25] and also [26, 7].

The scalar curvature of a surface which evolves under the Ricci flow satisfies the heat equation with a potential on that surface. In the same spirit, we expect to find geometric quantities on M that obey (1.1)–(1.2). The gradient estimates in this paper would then lead to new knowledge about the behavior of the metric $g(x,t)$ under the Ricci flow. In particular, we believe that our results will be helpful in classifying ancient solutions of (1.1). L. Ni's work [24] offers yet another way to use the Ricci flow combined with the heat equation to study the evolution of $g(x,t)$.

Subsection 2.2 discusses space-only gradient estimates for system (1.1)–(1.2). The first predecessor of these results was obtained by R. Hamilton in the paper [15]. It applies to the case where M is a closed manifold, the metric $g(x,t)$ is independent of t, and equation (1.1) is not in the picture. New versions of R. Hamilton's result were proposed in [31, 36, 5, 6]. The beginning of Section 2 describes them thoroughly. For related work done by probabilistic methods, one should consult [18, Chapter 5] and [2]. Theorem 2.2 states a space-only gradient estimate for (1.1)–(1.2). It is a result of local nature.

Subsection 2.3 deals with space-time gradient estimates for (1.1)–(1.2). Our results resemble the Li-Yau inequalities from the paper [22]; see also [27, Chapter IV]. More precisely, the solution $u(x,t)$ of equation (1.2) satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}, \qquad x \in M,\ t \in (0,T], \tag{1.3}$$

if M is a closed manifold with nonnegative Ricci curvature, the metric $g(x,t)$ does not depend on t, and (1.1) is not assumed. Here, ∇ stands for the gradient, the subscript t denotes the derivative in t, and n is the dimension of M. This result goes back to [22] and constitutes the simplest Li-Yau inequality. It opened new possibilities for the comparison of the values of solutions of (1.2) at different points and led to important Gaussian bounds in heat kernel analysis. Integrating the above estimate along a space-time curve yields a Harnack inequality. A precursory form of (1.3) appeared in [3]. Many variants of (1.3) now exist in the literature; see, e.g., [20, 11, 4, 21]. R. Hamilton proved one in [15] which further extended our ability to compare the values of solutions of (1.2). Li-Yau inequalities served as prototypes for many estimates connected to geometric flows. The list of relevant references includes but is not limited to [8, 17, 10]. In particular, the Li-Yau-type inequality for the Ricci flow became one of the central tools in classifying ancient solutions to the flow as detailed in [12, Chapter 9]. Analogous results played a significant part in the study of Kähler manifolds; see [9, Chapter 2]. Our Theorems 2.7 and 2.9 establish space-time gradient estimates for (1.1)–(1.2). As an application, we lay down two Harnack inequalities for (1.1)–(1.2). They help compare the values of a solution at different points. We are also hopeful that the techniques in Subsection 2.3 will lead to the discovery of new informative Li-Yau-type inequalities related to the Ricci flow and other geometric flows. Our investigation of (1.1)–(1.2) would then be a model for the proof of such inequalities.

In Section 3, we consider the case where M is a compact manifold and $\partial M \neq \emptyset$. We impose the boundary

condition on the Ricci flow (1.1) by demanding that the second fundamental form $\mathrm{II}(x,t)$ of the boundary with respect to $g(x,t)$ satisfy

$$\mathrm{II}(x,t) = \lambda(t) g(x,t), \qquad x \in \partial M,\ t \in [0,T], \tag{1.4}$$

for some nonnegative function $\lambda(t)$ defined on $[0,T]$. Thus, ∂M must remain convex and umbilic[1] for all $t \in [0,T]$. We then assume $u(x,t)$ solves the heat equation (1.2) and satisfies the Neumann boundary condition

$$\frac{\partial}{\partial \nu} u(x,t) = 0, \qquad x \in \partial M,\ t \in [0,T]. \tag{1.5}$$

The outward unit normal $\frac{\partial}{\partial \nu}$ is determined by the metric $g(x,t)$ and, therefore, depends on the parameter t.

The Ricci flow on manifolds with boundary is not yet deeply understood. We remind the reader that equation (1.1) fails to be strictly parabolic. As a consequence, it is not even clear how to impose the boundary conditions on (1.1) to obtain a well-posed problem. Progress in this direction was made by Y. Shen in the paper [28]. He proposed to consider the Ricci flow on a manifold with boundary assuming formula (1.4) holds with $\lambda(t)$ identically equal to a constant. Furthermore, he managed to prove the short-time existence of solutions to the flow in this case. The work [13] continues the investigation of problem (1.1) subject to (1.4) with $\lambda(t)$ equal to a constant. It also contains a complete set of references on the subject. In the present paper, we consider a more general situation by allowing $\lambda(t)$ to depend on the parameter t nontrivially. Note that Y. Shen's method of proving the short-time existence applies to this case, as well.

Subsection 3.1 ponders on the geometric meaning of the function $\lambda(t)$. We explain why it is beneficial to let $\lambda(t)$ depend on t. The discussion is rather informal. Subsection 3.2 provides gradient estimates for system (1.1)–(1.2) subject to the boundary conditions (1.4)–(1.5). Theorems 3.1 and 3.4 state versions of inequalities from Theorems 2.4 and 2.9. Related work was done in [22, 34, 4, 25]. Note that Theorem 3.1 appears to be new in the case where ∂M is nonempty even if $g(x,t)$ is independent of t (see Remark 3.3 for the details). At the same time, the proof is not particularly complicated.

Theorems 3.1 and 3.4 are likely to have applications similar to those of Theorems 2.4 and 2.9. We hope that the material in Section 3 will help shed light on the behavior of the Ricci flow on manifolds with boundary. Last but not least, our results may serve as a model for the investigation of problems similar to (1.1)–(1.2)–(1.4)–(1.5). For example, is it natural to look at the Ricci flow subject to (1.4) combined with the conjugate heat equation. As we previously explained, such problems were actively studied on manifolds without boundary, but the case where ∂M is nonempty remains unexplored.

Note. After this paper was completed, we became aware that space-time gradient estimates for (1.1)–(1.2) were researched independently by Shiping Liu in *Gradient estimates for solutions of the heat equation under Ricci Flow*, Pacific Journal of Mathematics 243 (2009) 165–180, and Jun Sun in *Gradient estimates for positive solutions of the heat equation under geometric flow*, preprint. The results of those works are not identical to ours.

2 Manifolds without boundary

Our goal is to investigate the Ricci flow combined with the heat equation. The present section establishes space-only and space-time gradient estimates in this context.

2.1 The setup

Suppose M is a connected, oriented, smooth, n-dimensional manifold without boundary. Some of the results in this section, but not all of them, concern the case where M is compact. Given $T > 0$, assume $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a complete solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \qquad x \in M,\ t \in [0,T]. \tag{2.1}$$

[1] There is ambiguity in the literature as to the use of the term "umbilic" in this context. See the discussion in [13].

Suppose a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfies the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad x \in M,\ t \in [0,T]. \tag{2.2}$$

Here, Δ stands for the Laplacian given by $g(x,t)$. In what follows, we will use the notation ∇ and $|\cdot|$ for the gradient and the norm with respect to $g(x,t)$. It is clear that Δ, ∇, and $|\cdot|$ all depend on $t \in [0,T]$. We will write XY for the scalar product of the vectors X and Y with respect to $g(x,t)$.

Subsection 2.2 offers space-only gradient estimates for $u(x,t)$. These results require that $u(x,t)$ be a bounded function. A local space-only gradient estimate for solutions of (2.2) was originally proved in the paper [31] in the situation where $g(x,t)$ did not depend on $t \in [0,T]$ and (2.1) was not in the picture. It was further generalized in [36] to hold in the case of the backward Ricci flow combined with the heat equation. Our Theorem 2.2 constitutes a version of this result for $u(x,t)$. A global space-only gradient estimate for solutions of (2.2) was originally established in [15] with $g(x,t)$ independent of $t \in [0,T]$ and (2.1) not assumed. It is now known to hold in the cases of both the backward Ricci flow and the Ricci flow combined with the heat equation; see [36, 5, 6]. We restate it in Theorem 2.4 for the completeness of our exposition. Subsection 2.3 contains Li-Yau-type estimates for (2.1)–(2.2). As applications, we obtain two Harnack inequalities.

The results in this section prevail, with obvious modifications, if the function $u(x,t)$ is defined on $M\times(0,T]$ instead of $M\times[0,T]$. In order to see this, it suffices to replace $u(x,t)$ and $g(x,t)$ with $u(x,t+\epsilon)$ and $g(x,t+\epsilon)$ for a sufficiently small $\epsilon > 0$, apply the corresponding formula, and then let ϵ go to 0. We thus justify, for example, the application of the theorems in Subsection 2.3 to heat-kernel-type functions.

Two more pieces of notation should be introduced at this point. Let us fix $x_0 \in M$ and $\rho > 0$. We write $\mathrm{dist}(\chi, x_0, t)$ for the distance between $\chi \in M$ and x_0 with respect to the metric $g(x,t)$. The notation $B_{\rho,T}$ stands for the set $\{(\chi,t) \in M \times [0,T] \,|\, \mathrm{dist}(\chi,x_0,t) < \rho\}$. We point out that Theorems 2.2 and 2.7 still hold if $u(x,t)$ is defined on $B_{\rho,T}$ instead of $M \times [0,T]$ and satisfies the heat equation in $B_{\rho,T}$.

The proofs in this section will often involve local computations. Therefore, we assume a coordinate system $\{x_1,\dots,x_n\}$ is fixed in a neighborhood of every point $x \in M$. The notation R_{ij} refers to the corresponding components of the Ricci tensor. In order to facilitate the computations, we often implicitly assume that $\{x_1,\dots,x_n\}$ are normal coordinates at $x \in M$ with respect to the appropriate metric. We use the standard shorthand: Given a real-valued function f on the manifold M, the notation f_i stands for $\frac{\partial f}{\partial x_i}$, the notation f_{ij} refers to the Hessian of f applied to $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$, and f_{ijk} is the third covariant derivative applied to $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_j}$, and $\frac{\partial}{\partial x_k}$. The subscript t designates the differentiation in $t \in [0,T]$.

The proofs of Theorems 2.2 and 2.7 will involve a cut-off function on $B_{\rho,T}$. The construction of this function will rely on the basic analytical result stated in the following lemma. This result is well-known. For example, it was previously used in the proofs of Theorems 2.3 and 3.1 in [36]; see also [27, Chapter IV] and [31].

Lemma 2.1. *Given $\tau \in (0,T]$, there exists a smooth function $\bar\Psi : [0,\infty)\times[0,T] \to \mathbb{R}$ satisfying the following requirements:*

1. *The support of $\bar\Psi(r,t)$ is a subset of $[0,\rho]\times[0,T]$, and $0 \le \bar\Psi(r,t) \le 1$ in $[0,\rho]\times[0,T]$.*
2. *The equalities $\bar\Psi(r,t) = 1$ and $\frac{\partial\bar\Psi}{\partial r}(r,t) = 0$ hold in $\left[0,\frac{\rho}{2}\right]\times[\tau,T]$ and $\left[0,\frac{\rho}{2}\right]\times[0,T]$, respectively.*
3. *The estimate $\left|\frac{\partial\bar\Psi}{\partial t}\right| \le \frac{\bar C \bar\Psi^{\frac{1}{2}}}{\tau}$ is satisfied on $[0,\infty)\times[0,T]$ for some $\bar C > 0$, and $\bar\Psi(r,0) = 0$ for all $r \in [0,\infty)$.*
4. *The inequalities $-\frac{C_a\bar\Psi^a}{\rho} \le \frac{\partial\bar\Psi}{\partial r} \le 0$ and $\left|\frac{\partial^2\bar\Psi}{\partial r^2}\right| \le \frac{C_a\bar\Psi^a}{\rho^2}$ hold on $[0,\infty)\times[0,T]$ for every $a \in (0,1)$ with some constant C_a dependent on a.*

2.2 Space-only gradient estimates

Let us begin by stating the local space-only gradient estimate.

Theorem 2.2. *Suppose $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a complete solution to the Ricci flow (2.1). Assume that $|\operatorname{Ric}(x,t)| \le k$ for some $k > 0$ and all $(x,t) \in B_{\rho,T}$. Suppose $u : M \times [0,T] \to \mathbb{R}$ is a smooth positive function solving the heat equation (2.2). If $u(x,t) \le A$ for some $A > 0$ and all $(x,t) \in B_{\rho,T}$, then there exists a constant C that depends only on the dimension of M and satisfies*

$$\frac{|\nabla u|}{u} \le C\left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{k}\right)\left(1 + \log\frac{A}{u}\right) \tag{2.3}$$

for all $(x,t) \in B_{\frac{\rho}{2},T}$ with $t \neq 0$.

We will now establish a lemma of computational character. It will play a significant part in the proof of Theorem 2.2.

Lemma 2.3. *Let $\big(M, g(x,t)\big)_{t\in[0,T]}$ be a complete solution to the Ricci flow (2.1). Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfying the heat equation (2.2). Assume that $u(x,t) \le 1$ for all $(x,t) \in B_{\rho,T}$. Let $f = \log u$ and $w = \frac{|\nabla f|^2}{(1-f)^2}$. Then the inequality*

$$\left(\Delta - \frac{\partial}{\partial t}\right) w \ge \frac{2f}{1-f}\nabla f \nabla w + 2(1-f)w^2$$

holds in $B_{\rho,T}$.

Proof. A direct computation demonstrates that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) w = &\sum_{i,j=1}^{n}\left(\frac{2f_{ij}^2}{(1-f)^2} + 8\frac{f_i f_{ij} f_j}{(1-f)^3} - 4\frac{f_i f_j f_{ij}}{(1-f)^2}\right)\\ &+ 6\frac{|\nabla f|^4}{(1-f)^4} - 2\frac{|\nabla f|^4}{(1-f)^3}\end{aligned}$$

and

$$4\sum_{i,j=1}^{n}\frac{f_i f_{ij} f_j}{(1-f)^3} = 2\frac{\nabla f \nabla w}{(1-f)} - 4\frac{|\nabla f|^4}{(1-f)^4}$$

at every point $(x,t) \in B_{\rho,T}$; cf. [31, 36]. Using these formulas, we conclude that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) w &= \sum_{i,j=1}^{n}\left(\frac{2f_{ij}^2}{(1-f)^2} + 4\frac{f_i f_{ij} f_j}{(1-f)^3} - 4\frac{f_i f_j f_{ij}}{(1-f)^2}\right)\\ &\quad + 2\frac{|\nabla f|^4}{(1-f)^4} + 2\frac{\nabla f \nabla w}{(1-f)} - 2\frac{|\nabla f|^4}{(1-f)^3}\\ &= 2\sum_{i,j=1}^{n}\left(\frac{f_{ij}}{1-f} + \frac{f_i f_j}{(1-f)^2}\right)^2\\ &\quad + 2\frac{\nabla f \nabla w}{(1-f)} + 2\frac{|\nabla f|^4}{(1-f)^3} - 2\nabla f \nabla w\\ &\ge \frac{2f}{1-f}\nabla f \nabla w + 2(1-f)w^2\end{aligned}$$

at $(x,t) \in B_{\rho,T}$. ∎

The preparations required to prove Theorem 2.2 are now completed. Note that we will also make use of arguments from the paper [36].

Proof of Theorem 2.2. Without loss of generality, we can assume $A = 1$. If this is not the case, one should just carry out the proof replacing $u(x,t)$ with $\frac{u(x,t)}{A}$. Let us pick a number $\tau \in (0,T]$ and fix a function $\bar{\Psi}(r,t)$ satisfying the conditions of Lemma 2.1. We will establish (2.3) at (x,τ) for all x such that $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. Because τ is chosen arbitrarily, the assertion of the theorem will immediately follow.

Define $\Psi : M \times [0,T] \to \mathbb{R}$ by the formula

$$\Psi(x,t) = \bar{\Psi}(\operatorname{dist}(x,x_0,t),t).$$

It is easy to see that $\Psi(x,t)$ is supported in the closure of $B_{\rho,T}$. This function is smooth at $(x',t') \in M \times [0,T]$ whenever $x' \neq x_0$ and x' is not in the cut locus of x_0 with respect to the metric $g(x,t')$. We will employ the notation $f = \log u$ and $w = \frac{|\nabla f|^2}{(1-f)^2}$ introduced in Lemma 2.3. It will also be convenient for us to write β instead of $-\frac{2f}{1-f}\nabla f$. Our strategy is to estimate $\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w)$ and scrutinize the produced formula at a point where Ψw attains its maximum. The desired result will then follow.

We use Lemma 2.3 to conclude that

$$\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w) \geq \Psi\left(-\beta\nabla w + 2(1-f)w^2\right) + (\Delta\Psi)w + 2\nabla\Psi\nabla w - \Psi_t w$$

in the portion of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth. This implies

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi w) \geq & -\beta\nabla(\Psi w) + \frac{2}{\Psi}\nabla\Psi\nabla(\Psi w) + 2\Psi(1-f)w^2 \\ & + w\beta\nabla\Psi - 2\frac{|\nabla\Psi|^2}{\Psi}w + (\Delta\Psi)w - \Psi_t w. \end{aligned} \tag{2.4}$$

The latter inequality holds in the part of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth and nonzero. Now let (x_1,t_1) be a maximum point for Ψw in the closure of $B_{\rho,T}$. If $(\Psi w)(x_1,t_1)$ is equal to 0, then $(\Psi w)(x,\tau) = w(x,\tau) = 0$ for all $x \in M$ such that $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. This yields $\nabla u(x,\tau) = 0$, and the estimate (2.3) becomes obvious at (x,τ). Thus, it suffices to consider the case where $(\Psi w)(x_1,t_1) > 0$. In particular, (x_1,t_1) must be in $B_{\rho,T}$, and t_1 must be strictly positive.

A standard argument due to E. Calabi (see, for example, [27, page 21]) enables us to assume that $\Psi(x,t)$ is smooth at (x_1,t_1). Because (x_1,t_1) is a maximum point, the formulas $\Delta(\Psi w)(x_1,t_1) \leq 0$, $\nabla(\Psi w)(x_1,t_1) = 0$, and $(\Psi w)_t(x_1,t_1) \geq 0$ hold true. Together with (2.4), they yield

$$2\Psi(1-f)w^2 \leq -w\beta\nabla\Psi + 2\frac{|\nabla\Psi|^2}{\Psi}w - (\Delta\Psi)w + \Psi_t w \tag{2.5}$$

at (x_1,t_1). We will now estimate every term in the right-hand side. This will lead us to the desired result.

A series of computations imply that

$$\begin{aligned}|w\beta\nabla\Psi| &\leq \Psi(1-f)w^2 + \frac{c_1 f^4}{\rho^4(1-f)^3}, \\ \frac{|\nabla\Psi|^2}{\Psi}w &\leq \frac{1}{8}\Psi w^2 + \frac{c_1}{\rho^4}, \\ -(\Delta\Psi)w &\leq \frac{1}{8}\Psi w^2 + \frac{c_1}{\rho^4} + c_1 k^2\end{aligned}$$

at (x_1,t_1) for some constant $c_1 > 0$; see [31, 36]. Here, we have used the inequality for the weighted arithmetic mean and the weighted geometric mean, as well as the properties of the function $\bar{\Psi}(r,t)$ given by Lemma 2.1. Our next mission is to find a suitable bound for $(\Psi_t w)(x_1,t_1)$.

It is clear that

$$\begin{aligned}(\Psi_t w)(x_1,t_1) = & \frac{\partial\bar{\Psi}}{\partial t}(\operatorname{dist}(x_1,x_0,t_1),t_1)w(x_1,t_1) \\ & + \frac{\partial\bar{\Psi}}{\partial r}(\operatorname{dist}(x_1,x_0,t_1),t_1)\left(\frac{\partial}{\partial t}\operatorname{dist}(x_1,x_0,t_1)\right)w(x_1,t_1).\end{aligned} \tag{2.6}$$

We also observe that

$$\left|\frac{\partial \bar{\Psi}}{\partial t}(\mathrm{dist}(x_1, x_0, t_1), t_1)\right| w(x_1, t_1) \leq \frac{1}{16}\left(\Psi w^2\right)(x_1, t_1) + \frac{c_2}{\tau^2}$$

for a positive constant c_2. Because the function $\bar{\Psi}(r, t)$ satisfies the conditions listed in Lemma 2.1, the inequality

$$\left|\frac{\partial \bar{\Psi}}{\partial r}(\mathrm{dist}(x_1, x_0, t_1), t_1)\right| \leq \frac{C_{\frac{1}{2}}}{\rho}\,\Psi^{\frac{1}{2}}(x_1, t_1) \tag{2.7}$$

holds with $C_{\frac{1}{2}} > 0$. It remains to estimate the derivative of the distance. Utilizing the assumptions of the theorem, we conclude that

$$\begin{aligned}\left|\frac{\partial}{\partial t}\,\mathrm{dist}(x_1, x_0, t_1)\right| &\leq \sup \int_0^{\mathrm{dist}(x_1, x_0, t_1)} \left|\mathrm{Ric}\left(\frac{d}{ds}\zeta(s), \frac{d}{ds}\zeta(s)\right)\right| ds \\ &\leq k\,\mathrm{dist}(x_1, x_0, t_1) \leq k\rho. \end{aligned} \tag{2.8}$$

In this particular formula, Ric designates the Ricci curvature of $g(x, t_1)$. The supremum is taken over all the minimal geodesics $\zeta(s)$, with respect to $g(x, t_1)$, that connect x_0 to x_1 and are parametrized by arclength; see, e.g., [12, Proof of Lemma 8.28]. It now becomes clear that

$$\Psi_t w \leq \frac{1}{16}\Psi w^2 + \frac{c_2}{\tau^2} + C_{\frac{1}{2}} k w \Psi^{\frac{1}{2}} \leq \frac{1}{8}\Psi w^2 + \frac{c_2}{\tau^2} + c_3 k^2$$

at (x_1, t_1) for some $c_3 > 0$. We have thus found estimates for every term in the right-hand side of (2.5). We will combine them all, and the assertion of the theorem will shortly follow.

Given the preceding considerations, formula (2.5) implies

$$\Psi(1-f)w^2 \leq \frac{c_4 f^4}{\rho^4(1-f)^3} + \frac{1}{2}\,\Psi w^2 + \frac{c_4}{\rho^4} + \frac{c_4}{\tau^2} + c_4 k^2$$

at the point (x_1, t_1). The constant c_4 here equals $\max\{3c_1, c_2, c_1 + c_3\}$. Since $f(x,t) \leq 0$ and $\frac{f^4}{(1-f)^4} \leq 1$, we can conclude that

$$\begin{aligned}\Psi w^2 &\leq \frac{c_4 f^4}{\rho^4(1-f)^4} + \frac{1}{2}\Psi w^2 + \frac{c_4}{\rho^4} + \frac{c_4}{\tau^2} + c_4 k^2, \\ \Psi^2 w^2 \leq \Psi w^2 &\leq \frac{4c_4}{\rho^4} + \frac{2c_4}{\tau^2} + 2c_4 k^2\end{aligned}$$

at (x_1, t_1). Because $\Psi(x, \tau) = 1$ when $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$, the estimate

$$w(x, \tau) = (\Psi w)(x, \tau) \leq (\Psi w)(x_1, t_1) \leq \frac{C^2}{\rho^2} + \frac{C^2}{\tau} + C^2 k$$

holds with $C = \sqrt{2\sqrt{c_4}}$ for all $x \in M$ such that $\mathrm{dist}(x, x_0, \tau) < \frac{\rho}{2}$. Recalling the definition of $w(x,t)$ and the fact that $\tau \in (0, T]$ was chosen arbitrarily, we obtain the inequality

$$\frac{|\nabla f(x,t)|}{1 - f(x,t)} \leq C\left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{k}\right)$$

for $(x, t) \in B_{\frac{\rho}{2}, T}$ provided $t \neq 0$. The assertion of the theorem follows by an elementary computation. ☐

Our next step is to assume M is compact and state a global gradient estimate for the function $u(x,t)$. This result was previously established in [36, 6]. We restate it here for the completeness of our exposition. Moreover, we believe it is appropriate to present the proof, which is quite short. A computation from this proof will be used in Section 3.

Theorem 2.4 (Q. Zhang [36], X. Cao and R. Hamilton [6])**.** *Suppose the manifold M is compact, and let $\big(M, g(x,t)\big)_{t\in[0,T]}$ be a solution to the Ricci flow* (2.1)*. Assume a smooth positive function $u : M\times[0,T]\to\mathbb{R}$ satisfies the heat equation* (2.2)*. Then the estimate*

$$\frac{|\nabla u|}{u} \le \sqrt{\frac{1}{t}\log\frac{A}{u}}\,, \qquad x\in M,\ t\in(0,T], \tag{2.9}$$

holds with $A = \sup_M u(x,0)$.

Remark 2.5. The maximum principle implies that A is actually equal to $\sup_{M\times[0,T]} u(x,t)$. This explains why the right-hand side of (2.9) is well-defined.

Proof. Consider the function $P = t\frac{|\nabla u|^2}{u} - u\log\frac{A}{u}$ on the set $M\times[0,T]$. It is clear that $P(x,0)$ is nonpositive for every $x\in M$. A computation shows that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)P &= t\left(\Delta - \frac{\partial}{\partial t}\right)\left(\frac{|\nabla u|^2}{u}\right)\\ &= 2\frac{t}{u}\sum_{i,j=1}^{n}\left(u_{ij} - \frac{u_i u_j}{u}\right)^2 \ge 0, \qquad x\in M,\ t\in[0,T].\end{aligned}$$

In accordance with the maximum principle, this implies $P(x,t)$ is nonpositive for all $(x,t)\in M\times[0,T]$. The desired assertion follows immediately. ☐

2.3 Space-time gradient estimates

This subsection establishes Li-Yau-type inequalities for system (2.1)–(2.2). We will obtain a local and a global estimate. The following lemma will be important to our considerations; cf. Lemma 1 in [27, Chapter IV]. It will also reoccur in Section 3.

Lemma 2.6. *Suppose $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a complete solution to the Ricci flow* (2.1)*. Assume that $-k_1 g(x,t) \le \mathrm{Ric}(x,t) \le k_2 g(x,t)$ for some $k_1, k_2 > 0$ and all $(x,t)\in B_{\rho,T}$. Suppose $u : M\times[0,T]\to\mathbb{R}$ is a smooth positive function satisfying the heat equation* (2.2)*. Given $\alpha\ge 1$, define $f = \log u$ and $F = t\left(|\nabla f|^2 - \alpha f_t\right)$. The estimate*

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)F \ge &-2\nabla f\nabla F\\ &+\frac{2a\alpha t}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right)\\ &-2k_1\alpha t|\nabla f|^2 - \frac{\alpha t n}{2b}\max\left\{k_1^2, k_2^2\right\}, \qquad (x,t)\in B_{\rho,T},\end{aligned} \tag{2.10}$$

holds for any $a, b > 0$ such that $a + b = \frac{1}{\alpha}$.

Proof. We begin by finding a convenient bound on ΔF. Observe that

$$\Delta F = t\left(2\sum_{i,j=1}^{n}\left(f_{ij}^2 + 2f_j f_{jii}\right) - \alpha\Delta(f_t)\right), \qquad x\in M,\ t\in[0,T].$$

Our assumption on the Ricci curvature of M implies the inequality

$$\begin{aligned}\sum_{i,j=1}^{n} f_j f_{jii} &= \sum_{i,j=1}^{n}\left(f_j f_{iij} + R_{ij} f_i f_j\right)\\ &= \nabla f\nabla(\Delta f) + \mathrm{Ric}(\nabla f, \nabla f) \ge \nabla f\nabla(\Delta f) - k_1|\nabla f|^2\end{aligned}$$

at an arbitrary point $(x,t) \in B_{\rho,T}$. Using (2.1), we can show that

$$\Delta(f_t) = (\Delta f)_t - 2\sum_{i,j=1}^n R_{ij} f_{ij}.$$

Consequently, the estimate

$$\Delta F \geq t\left(2\sum_{i,j=1}^n \left(f_{ij}^2 + 2\alpha R_{ij} f_{ij}\right) + 2\nabla f \nabla(\Delta f) - 2k_1|\nabla f|^2 - \alpha(\Delta f)_t\right)$$

holds at $(x,t) \in B_{\rho,T}$. Our next step is to find a suitable bound on those terms in the right-hand side that involve f_{ij}. We do so by completing the square. More specifically, observe that

$$\begin{aligned}
\sum_{i,j=1}^n \left(f_{ij}^2 + \alpha R_{ij} f_{ij}\right) &= \sum_{i,j=1}^n \left((a\alpha + b\alpha) f_{ij}^2 + \alpha R_{ij} f_{ij}\right) \\
&= \sum_{i,j=1}^n \left(a\alpha f_{ij}^2 + \alpha\left(\sqrt{b}\, f_{ij} + \frac{R_{ij}}{2\sqrt{b}}\right)^2 - \frac{\alpha}{4b} R_{ij}^2\right) \\
&\geq \sum_{i,j=1}^n \left(a\alpha f_{ij}^2 - \frac{\alpha}{4b} R_{ij}^2\right)
\end{aligned}$$

at $(x,t) \in B_{\rho,T}$ for any $a, b > 0$ such that $a + b = \frac{1}{\alpha}$. Employing the standard inequality

$$\sum_{i,j=1}^n f_{ij}^2 \geq \frac{(\Delta f)^2}{n}$$

and the assumptions of the lemma, we obtain the estimate

$$\sum_{i,j=1}^n \left(f_{ij}^2 + \alpha R_{ij} f_{ij}\right) \geq \frac{a\alpha}{n}(\Delta f)^2 - \frac{\alpha n}{4b} \max\left\{k_1^2, k_2^2\right\}, \qquad (x,t) \in B_{\rho,T}.$$

It is easy to conclude that

$$\begin{aligned}
\Delta F &\geq t\left(\frac{2a\alpha}{n}(\Delta f)^2 + 2\nabla f\nabla(\Delta f) - 2k_1|\nabla f|^2 - \alpha(\Delta f)_t - \frac{\alpha n}{2b} \max\left\{k_1^2, k_2^2\right\}\right) \\
&= \frac{2a\alpha t}{n}\left(f_t - |\nabla f|^2\right)^2 + 2t\nabla f\nabla\left(f_t - |\nabla f|^2\right) \\
&\quad - 2k_1 t|\nabla f|^2 - \alpha t\left(f_t - |\nabla f|^2\right)_t - \frac{\alpha t n}{2b} \max\left\{k_1^2, k_2^2\right\} \qquad (2.11)
\end{aligned}$$

in the set $B_{\rho,T}$.

Formula (2.11) provides us with a convenient bound on ΔF. Let us now include the derivative of F in $t \in [0,T]$ into our considerations. One easily computes

$$\frac{\partial F}{\partial t} = |\nabla f|^2 - \alpha f_t + t\left(|\nabla f|^2 - \alpha f_t\right)_t.$$

Subtracting this from (2.11), we see that the inequality

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) F \geq\ & \frac{2a\alpha t}{n}\left(f_t - |\nabla f|^2\right)^2 + 2t\nabla f\nabla\left(f_t - |\nabla f|^2\right) - 2k_1 t|\nabla f|^2 \\
& - \frac{\alpha t n}{2b} \max\left\{k_1^2, k_2^2\right\} - \left(|\nabla f|^2 - \alpha f_t\right) + (\alpha - 1)t\left(|\nabla f|^2\right)_t
\end{aligned}$$

holds in the set $B_{\rho,T}$. In order to arrive to (2.10) from here, we need to estimate $\left(|\nabla f|^2\right)_t$. The Ricci flow equation (2.1) and the assumptions of the lemma imply

$$\left(|\nabla f|^2\right)_t = 2\nabla f\nabla(f_t) + 2\operatorname{Ric}(\nabla f,\nabla f) \geq 2\nabla f\nabla(f_t) - 2k_1|\nabla f|^2$$

at $(x,t)\in B_{\rho,T}$. As a consequence,

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)F \geq{}& \frac{2a\alpha t}{n}\left(f_t - |\nabla f|^2\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right)\\ &- \frac{\alpha t n}{2b}\max\left\{k_1^2, k_2^2\right\} - 2t\nabla f\nabla\left(|\nabla f|^2 - \alpha f_t\right) - 2k_1\alpha t|\nabla f|^2\end{aligned}$$

in $B_{\rho,T}$. The desired assertion follows immediately. □

With Lemma 2.6 at hand, we are ready to establish the local space-time gradient estimate. We will also make use of arguments from the proof of Theorem 4.2 in [27, Chapter IV]. Recall that n designates the dimension of M.

Theorem 2.7. *Let $\left(M, g(x,t)\right)_{t\in[0,T]}$ be a complete solution to the Ricci flow (2.1). Suppose $-k_1 g(x,t) \leq \operatorname{Ric}(x,t) \leq k_2 g(x,t)$ for some $k_1, k_2 > 0$ and all $(x,t)\in B_{\rho,T}$. Consider a smooth positive function $u : M\times[0,T]\to\mathbb{R}$ solving the heat equation (2.2). There exists a constant C' that depends only on the dimension of M and satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq C'\alpha^2\left(\frac{\alpha^2}{\rho^2(\alpha-1)} + \frac{1}{t} + \max\{k_1,k_2\}\right) + \frac{nk_1\alpha^3}{\alpha-1} \tag{2.12}$$

for all $\alpha > 1$ and all $(x,t)\in B_{\frac{\rho}{2},T}$ with $t\neq 0$.

Proof. We preserve the notation $f = \log u$ and $F = t\left(|\nabla f|^2 - \alpha f_t\right)$ from Lemma 2.6. Our strategy in this proof will be similar to that in the proof of Theorem 2.2. The role of the function $w(x,t)$ now goes to the function $F(x,t)$.

Let us pick $\tau\in(0,T]$ and fix $\bar\Psi(r,t)$ satisfying the conditions of Lemma 2.1. Define $\Psi : M\times[0,T]\to\mathbb{R}$ by setting

$$\Psi(x,t) = \bar\Psi(\operatorname{dist}(x,x_0,t),t).$$

We will establish (2.12) at (x,τ) for $x\in M$ such that $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. This will complete the proof. Our plan is to estimate $\left(\frac{\partial}{\partial t} - \Delta\right)(\Psi F)$ and analyze the result at a point where the function ΨF attains its maximum. The required conclusion will follow therefrom.

Lemma 2.6 and some straightforward computations imply

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(\Psi F) \geq{}& -2\nabla f\nabla(\Psi F) + 2F\nabla f\nabla\Psi\\ &+ \left(\frac{2a\alpha t}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2 - \alpha f_t\right)\right)\Psi\\ &- \left(2k_1\alpha t|\nabla f|^2 + \frac{\alpha t n}{2b}\bar k^2\right)\Psi\\ &+ (\Delta\Psi)F + 2\frac{\nabla\Psi}{\Psi}\nabla(\Psi F) - 2\frac{|\nabla\Psi|^2}{\Psi}F - \frac{\partial\Psi}{\partial t}F\end{aligned} \tag{2.13}$$

with $\bar k = \max\{k_1,k_2\}$. This inequality holds in the part of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth and strictly positive. Let (x_1,t_1) be a maximum point for the function ΨF in the set $\{(x,t)\in M\times[0,\tau]\,|\,\operatorname{dist}(x,x_0,t)\leq\rho\}$. We may assume $(\Psi F)(x_1,t_1) > 0$ without loss of generality. Indeed, if this is not the case, then $F(x,\tau)\leq 0$ and (2.12) is evident at (x,τ) whenever $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. We may also assume that $\Psi(x,t)$ is smooth at (x_1,t_1) due to a standard trick explained, for example, in [27, page 21]. Since (x_1,t_1) is a maximum point,

the formulas $\Delta(\Psi F)(x_1,t_1)\le 0$, $\nabla(\Psi F)(x_1,t_1)=0$, and $(\Psi F)_t(x_1,t_1)\ge 0$ hold true. Combined with (2.13), they yield

$$\begin{aligned}
0 &\ge 2F\nabla f\nabla\Psi \\
&\quad + \left(\frac{2a\alpha t_1}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2-\alpha f_t\right) - 2k_1\alpha t_1|\nabla f|^2 - \frac{\alpha t_1 n}{2b}\,\bar k^2\right)\Psi \\
&\quad + (\Delta\Psi)F - 2\frac{|\nabla\Psi|^2}{\Psi}\,F - \frac{\partial\Psi}{\partial t}\,F
\end{aligned} \tag{2.14}$$

at (x_1,t_1). We will now use (2.14) to show that a certain quadratic expression in ΨF is nonpositive. The desired result will then follow.

Let us recall Lemma 2.1 and apply the Laplacian comparison theorem to conclude that

$$\begin{aligned}
-\frac{|\nabla\Psi|^2}{\Psi} &\ge -\frac{C_{\frac12}^2}{\rho^2}, \\
\Delta\Psi &\ge -\frac{C_{\frac12}}{\rho^2} - \frac{C_{\frac12}\Psi^{\frac12}}{\rho}\,(n-1)\sqrt{k_1}\,\coth\left(\sqrt{k_1}\,\rho\right) \ge -\frac{d_1}{\rho^2} - \frac{d_1\Psi^{\frac12}}{\rho}\,\sqrt{k_1}
\end{aligned}$$

at the point (x_1,t_1) with d_1 a positive constant depending on n. There exists $\bar C>0$ such that the inequality

$$-\frac{\partial\Psi}{\partial t} \ge -\frac{\bar C\Psi^{\frac12}}{\tau} - C_{\frac12}\bar k\Psi^{\frac12}$$

holds true; cf. (2.6), (2.7), and (2.8). Using these observations along with (2.14), we find the estimate

$$\begin{aligned}
0 &\ge -2F|\nabla f||\nabla\Psi| \\
&\quad + \left(\frac{2a\alpha t_1}{n}\left(|\nabla f|^2 - f_t\right)^2 - \left(|\nabla f|^2-\alpha f_t\right) - 2k_1\alpha t_1|\nabla f|^2 - \frac{\alpha t_1 n}{2b}\,\bar k^2\right)\Psi \\
&\quad + d_2\left(-\frac{1}{\rho^2} - \frac{\Psi^{\frac12}}{\rho}\,\sqrt{k_1} - \frac{\Psi^{\frac12}}{\tau} - \bar k\Psi^{\frac12}\right)F
\end{aligned}$$

at (x_1,t_1). Here, d_2 is equal to $\max\left\{3d_1, C_{\frac12}, 3C_{\frac12}^2, \bar C\right\}$. If one further multiplies by $t\Psi$ and makes a few elementary manipulations, one will obtain

$$\begin{aligned}
0 &\ge -2t_1F\frac{C_{\frac12}\Psi^{\frac32}}{\rho}\,|\nabla f| \\
&\quad + \frac{2t_1^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\,\bar k^2\Psi^2\right) \\
&\quad + d_2t_1\left(-\frac{1}{\rho^2} - \frac{\sqrt{k_1}}{\rho} - \frac{1}{\tau} - \bar k\right)(\Psi F) - \Psi F
\end{aligned} \tag{2.15}$$

at (x_1,t_1). Our next step is to estimate the first two terms in the right-hand side. In order to do so, we need a few auxiliary pieces of notation.

Define $y=\Psi|\nabla f|^2$ and $z=\Psi f_t$. It is clear that $y^{\frac12}(y-\alpha z)=\frac{\Psi^{\frac32}F|\nabla f|}{t}$ when $t\ne 0$, which yields

$$\begin{aligned}
&-2tF\frac{C_{\frac12}\Psi^{\frac32}}{\rho}\,|\nabla f| \\
&\quad + \frac{2t^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\,\bar k^2\Psi^2\right) \\
&\quad \ge \frac{2t^2}{n}\left(a\alpha(y-z)^2 - nk_1\alpha y - \frac{n^2\alpha}{4b}\,\bar k^2\Psi^2 - \frac{nC_{\frac12}}{\rho}\,y^{\frac12}(y-\alpha z)\right).
\end{aligned}$$

Let us observe that

$$(y-z)^2 = \frac{1}{\alpha^2}\,(y-\alpha z)^2 + \frac{(\alpha-1)^2}{\alpha^2}\,y^2 + \frac{2(\alpha-1)}{\alpha^2}\,y(y-\alpha z)$$

and plug this into the previous estimate. Regrouping the terms and applying the inequality $\kappa_1 v^2 - \kappa_2 v \geq -\frac{\kappa_2^2}{4\kappa_1}$ valid for $\kappa_1, \kappa_2 > 0$ and $v \in \mathbb{R}$, we obtain

$$\begin{aligned}
&-2tF\frac{C_{\frac{1}{2}}\Psi^{\frac{3}{2}}}{\rho}\,|\nabla f| \\
&\quad + \frac{2t^2}{n}\left(a\alpha\left(\Psi|\nabla f|^2 - \Psi f_t\right)^2 - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\,\bar{k}^2\Psi^2\right) \\
&\quad \geq \frac{2t^2}{n}\left(\frac{a}{\alpha}\,(y-\alpha z)^2 - \frac{n^2k_1^2\alpha^3}{4a(\alpha-1)^2}\right) \\
&\qquad - \frac{2t^2}{n}\left(\frac{n^2 d_2\alpha}{8a\rho^2(\alpha-1)}\,(y-\alpha z) + \frac{n^2\alpha}{4b}\,\bar{k}^2\Psi^2\right).
\end{aligned}$$

Because $t(y-\alpha z) = \Psi F$ by definition, (2.15) now implies

$$\begin{aligned}
0 &\geq \frac{2a}{n\alpha}\,(\Psi F)^2 + \left(-\frac{nd_2t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + 1 + \rho\sqrt{\bar{k}} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) - 1\right)(\Psi F) \\
&\quad - \frac{nk_1^2\alpha^3}{2a(\alpha-1)^2}\,t_1^2 - \frac{\alpha n}{2b}\,t_1^2\bar{k}^2\Psi^2 \\
&\geq \frac{2a}{n\alpha}\,(\Psi F)^2 + \left(-\frac{d_3t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) - 1\right)(\Psi F) \\
&\quad - \frac{nk_1^2\alpha^3}{2a(\alpha-1)^2}\,t_1^2 - \frac{\alpha n}{2b}\,t_1^2\bar{k}^2\Psi^2
\end{aligned}$$

at (x_1,t_1) with $d_3 = 4nd_2$. The expression in the last two lines is a polynomial in ΨF of degree 2. Consequently, in accordance with the quadratic formula,

$$\Psi F \leq \frac{n\alpha}{2a}\left(\frac{d_3t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) + 1 + \frac{k_1\alpha}{\alpha-1}\,t_1 + \sqrt{\frac{a}{b}}\,t_1\bar{k}\Psi\right)$$

at (x_1,t_1). We will now use this conclusion to obtain a bound on $F(x,\tau)$ for an appropriate range of $x \in M$.

Recall that $\Psi(x,\tau) = 1$ whenever $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. Besides, (x_1,t_1) is a maximum point for ΨF in the set $\{(x,t) \in M \times [0,\tau]\,|\,\operatorname{dist}(x,x_0,t) \leq \rho\}$. Hence

$$\begin{aligned}
F(x,\tau) &= (\Psi F)(x,\tau) \leq (\Psi F)(x_1,t_1) \\
&\leq \frac{n\alpha d_3\tau}{2a\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\bar{k}\right) + \frac{n\alpha}{2a} + \frac{nk_1\alpha^2}{2a(\alpha-1)}\,\tau + \frac{\alpha\tau n\bar{k}}{2}\sqrt{\frac{1}{ab}}
\end{aligned}$$

for all $x \in M$ such that $\operatorname{dist}(x,x_0,\tau) < \frac{\rho}{2}$. Since $\tau \in (0,T]$ was chosen arbitrarily, this formula implies

$$\begin{aligned}
\left(|\nabla f|^2 - \alpha f_t\right)(x,t) &\leq \frac{\alpha d_4}{a\rho^2}\left(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{t} + \rho^2\bar{k}\right) \\
&\quad + \frac{nk_1\alpha^2}{2a(\alpha-1)} + \frac{\alpha n\bar{k}}{2}\sqrt{\frac{1}{ab}}\,, \qquad (x,t) \in B_{\frac{\rho}{2},T},
\end{aligned}$$

with $d_4 = \max\{nd_3, n\}$ as long as $t \neq 0$. If we set $a = \frac{1}{2\alpha}$, note that $b = \frac{1}{\alpha} - a$, and define the constant C' appropriately, estimate (2.12) will follow by a straightforward computation. □

Remark 2.8. The value $\frac{1}{2\alpha}$ for the parameter a in the proof of the theorem might not be optimal. It is not unlikely that a different a will lead to a sharper estimate.

Let us now consider the case where the manifold M is compact. We will present a global estimate on $u(x,t)$ demanding that the Ricci curvature of M be nonnegative. A related inequality for (2.1)–(2.2) may be found in [14].

Theorem 2.9. *Suppose the manifold M is compact and $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a solution to the Ricci flow (2.1). Assume that $0 \le \mathrm{Ric}(x,t) \le kg(x,t)$ for some $k > 0$ and all $(x,t) \in M \times [0,T]$. Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfying the heat equation (2.2). The estimate*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le kn + \frac{n}{2t} \tag{2.16}$$

holds for all $(x,t) \in M \times (0,T]$.

Proof. As before, we write f instead of $\log u$. It will be convenient for us to denote $F_1 = t\left(|\nabla f|^2 - f_t\right)$. Fix $\tau \in (0,T]$ and choose a point $(x_0,t_0) \in M \times [0,\tau]$ where F_1 attains its maximum on $M \times [0,\tau]$. Our first step is to show that

$$F_1(x_0,t_0) \le t_0 kn + \frac{n}{2}\,. \tag{2.17}$$

The assertion of the theorem will follow therefrom.

If $t_0 = 0$, then $F_1(x,t_0)$ is equal to 0 for every $x \in M$ and estimate (2.17) becomes evident. Consequently, we can assume $t_0 > 0$ without loss of generality. Lemma 2.6 and our conditions on the Ricci curvature of M imply the inequality

$$\left(\Delta - \frac{\partial}{\partial t}\right) F_1 \ge -2\nabla f \nabla F_1 + \frac{2a}{n}\frac{F_1^2}{t_0} - \frac{F_1}{t_0} - \frac{t_0 n}{2(1-a)}\,k^2$$

for all $a \in (0,1)$ at the point (x_0,t_0). Now recall that F_1 attains its maximum at (x_0,t_0). This tells us that $\Delta F_1(x_0,t_0) \le 0$, $\frac{\partial}{\partial t}F_1(x_0,t_0) \ge 0$, and $\nabla F_1(x_0,t_0) = 0$. In consequence, the estimate

$$\frac{2a}{n}\frac{F_1^2}{t_0} - \frac{F_1}{t_0} - \frac{t_0 n}{2(1-a)}\,k^2 \le 0$$

holds at (x_0,t_0), and the quadratic formula yields

$$F_1(x_0,t_0) \le \frac{n}{4a}\left(1 + \sqrt{1 + \frac{4at_0^2}{1-a}\,k^2}\right).$$

The expression in the right-hand side is minimized in $a \in (0,1)$ when a is equal to $\frac{1+kt_0}{1+2kt_0}$. Plugging this value of a into the above inequality, we arrive at (2.17).

Only a simple argument is now needed to complete the proof. The fact that (x_0,t_0) is a maximum point for F_1 on $M \times [0,\tau]$ enables us to conclude that

$$F_1(x,\tau) \le F_1(x_0,t_0) \le t_0 kn + \frac{n}{2} \le \tau kn + \frac{n}{2}$$

for all $x \in M$. Therefore, the estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le kn + \frac{n}{2\tau}$$

holds at (x,τ). Because the number $\tau \in (0,T]$ can be chosen arbitrarily, the assertion of the theorem follows. □

Our last goal in this section is to state two Harnack inequalities for (2.1)–(2.2). These may be viewed as applications of Theorems 2.7 and 2.9; cf., for example, [27, Chapter IV]. One can find other Harnack

inequalities for (2.1)–(2.2) in the papers [14, 24]. We first introduce a piece of notation. Given $x_1, x_2 \in M$ and $t_1, t_2 \in (0,T)$ satisfying $t_1 < t_2$, define

$$\Gamma(x_1,t_1,x_2,t_2) = \inf \int_{t_1}^{t_2} \left|\frac{d}{dt}\gamma(t)\right|^2 dt.$$

The infimum is taken over the set $\Theta(x_1,t_1,x_2,t_2)$ of all the smooth paths $\gamma : [t_1,t_2] \to M$ that connect x_1 to x_2. We remind the reader that the norm $|\cdot|$ depends on t. Let us now present a lemma. It will be the key to the proof of our results.

Lemma 2.10. *Suppose $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a complete solution to the Ricci flow (2.1). Let $u : M \times [0,T] \to \mathbb{R}$ be a smooth positive function satisfying the heat equation (2.2). Define $f = \log u$ and assume that*

$$\frac{\partial f}{\partial t} \ge \frac{1}{A_1}\left(|\nabla f|^2 - A_2 - \frac{A_3}{t}\right), \qquad x \in M,\ t \in (0,T],$$

for some $A_1, A_2, A_3 > 0$. Then the inequality

$$u(x_2,t_2) \ge u(x_1,t_1)\left(\frac{t_2}{t_1}\right)^{-\frac{A_3}{A_1}} \exp\left(-\frac{A_1}{4}\Gamma(x_1,t_1,x_2,t_2) - \frac{A_2}{A_1}(t_2-t_1)\right)$$

holds for all $(x_1,t_1) \in M \times (0,T)$ and $(x_2,t_2) \in M \times (0,T)$ such that $t_1 < t_2$.

Proof. The method we use is rather traditional; see, for example, [27, Chapter IV] and [6]. Consider a path $\gamma(t) \in \Theta(x_1,t_1,x_2,t_2)$. We begin by computing

$$\begin{aligned}
\frac{d}{dt}f(\gamma(t),t) &= \nabla f(\gamma(t),t)\frac{d}{dt}\gamma(t) + \frac{\partial}{\partial s}f(\gamma(t),s)|_{s=t} \\
&\ge -|\nabla f(\gamma(t),t)|\left|\frac{d}{dt}\gamma(t)\right| + \frac{1}{A_1}\left(|\nabla f(\gamma(t),t)|^2 - A_2 - \frac{A_3}{t}\right) \\
&\ge -\frac{A_1}{4}\left|\frac{d}{dt}\gamma(t)\right|^2 - \frac{1}{A_1}\left(A_2 + \frac{A_3}{t}\right), \qquad t \in [t_1,t_2].
\end{aligned}$$

The last step is a consequence of the inequality $\kappa_1 v^2 - \kappa_2 v \ge -\frac{\kappa_2^2}{4\kappa_1}$ valid for $\kappa_1, \kappa_2 > 0$ and $v \in \mathbb{R}$. The above implies

$$\begin{aligned}
f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}f(\gamma(t),t)\,dt \\
&\ge -\frac{A_1}{4}\int_{t_1}^{t_2}\left|\frac{d}{dt}\gamma(t)\right|^2 dt - \frac{A_2}{A_1}(t_2-t_1) - \frac{A_3}{A_1}\ln\frac{t_2}{t_1}.
\end{aligned}$$

The assertion of the lemma follows by exponentiating. □

We are ready to formulate our Harnack inequalities for (2.1)–(2.2). The first one applies on noncompact manifolds. The second one does not, but it provides a more explicit estimate.

Theorem 2.11. *Let $\big(M, g(x,t)\big)_{t\in[0,T]}$ be a complete solution to the Ricci flow (2.1). Assume that $-k_1 g(x,t) \le \mathrm{Ric}(x,t) \le k_2 g(x,t)$ for some $k_1, k_2 > 0$ and all $(x,t) \in M \times [0,T]$. Suppose a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfies the heat equation (2.2). Given $\alpha > 1$, the estimate*

$$\begin{aligned}
u(x_2,t_2) &\ge u(x_1,t_1)\left(\frac{t_2}{t_1}\right)^{-C'\alpha} \\
&\qquad \exp\left(-\frac{\alpha}{4}\Gamma(x_1,t_1,x_2,t_2) - \left(C'\alpha\max\{k_1,k_2\} + \frac{nk_1\alpha^2}{\alpha-1}\right)(t_2-t_1)\right)
\end{aligned}$$

holds for all $(x_1,t_1) \in M \times (0,T)$ and $(x_2,t_2) \in M \times (0,T)$ such that $t_1 < t_2$. The constant C' comes from Theorem 2.7.

Proof. Letting ρ go to infinity in (2.12), we conclude that

$$\frac{u_t}{u} \geq \frac{1}{\alpha}\left(\frac{|\nabla u|^2}{u^2} - \frac{C'\alpha^2}{t} - \left(C'\alpha^2 \max\{k_1, k_2\} + \frac{nk_1\alpha^3}{\alpha - 1}\right)\right)$$

on $M \times (0, T]$. The desired assertion is now a consequence of Lemma 2.10. □

Theorem 2.12. *Suppose M is compact and $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a solution to the Ricci flow* (2.1)*. Assume that $0 \leq \mathrm{Ric}(x,t) \leq kg(x,t)$ for some $k > 0$ and all $(x,t) \in M \times [0,T]$. Consider a smooth positive function $u : M \times [0,T] \to \mathbb{R}$ satisfying the heat equation* (2.2)*. The estimate*

$$u(x_2, t_2) \geq u(x_1, t_1)\left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} \exp\left(-\frac{1}{4}\Gamma(x_1, t_1, x_2, t_2) - kn(t_2 - t_1)\right)$$

holds for all $(x_1, t_1) \in M \times (0,T)$ and $(x_2, t_2) \in M \times (0,T)$ as long as $t_1 < t_2$.

Proof. Theorem 2.9 implies

$$\frac{u_t}{u} \geq \frac{|\nabla u|^2}{u^2} - kn - \frac{n}{2t}, \qquad x \in M,\ t \in (0,T].$$

One may now use Lemma 2.10 to complete the proof. □

3 Manifolds with boundary

This section considers a compact manifold with boundary evolving under the Ricci flow and offers heat equation estimates on this manifold. We will present variants of Theorems 2.4 and 2.9. The proofs are largely based on the Hopf maximum principle.

3.1 The Ricci flow

Suppose M is a compact, connected, oriented, smooth manifold with nonempty boundary ∂M. Consider a Riemannian metric $g(x,t)$ on M that evolves under the Ricci flow. The parameter t runs through the interval $[0,T]$. We investigate the case where the boundary ∂M remains umbilic for all $t \in [0,T]$. More precisely, given a smooth nonnegative function $\lambda(t)$ on $[0,T]$, we assume that $\big(M, g(x,t)\big)_{t\in[0,T]}$ is a solution to the problem

$$\begin{aligned} &\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), && x \in M,\ t \in [0,T], \\ &\mathrm{II}(x,t) = \lambda(t) g(x,t), && x \in \partial M,\ t \in [0,T]. \end{aligned} \tag{3.1}$$

In the second line, $g(x,t)$ is understood to be restricted to the tangent bundle of ∂M. The notation $\mathrm{II}(x,t)$ here stands for the second fundamental form of ∂M with respect to $g(x,t)$. That is,

$$\mathrm{II}(X,Y) = \left(D_X \frac{\partial}{\partial \nu}\right) Y$$

if X and Y are tangent to the boundary at the same point. The letter D refers to the Levi-Civita connection corresponding to $g(x,t)$, and $\frac{\partial}{\partial \nu}$ is the outward unit normal vector field on ∂M with respect to $g(x,t)$.

We should explain that problem (3.1) has different geometric meanings for different choices of the function $\lambda(t)$. E.g., let us assume that $\lambda(t)$ is equal to the same constant λ_0 for all $t \in [0,T]$. The papers [28, 13] discuss this case in detail. Theorem 3 in [13] suggests that the Ricci flow (3.1), if normalized so as to preserve the volume of M, takes a sufficiently well-behaved Riemannian metric on M to a metric with totally geodesic boundary. An example of such an evolution is shown in Figure 1.

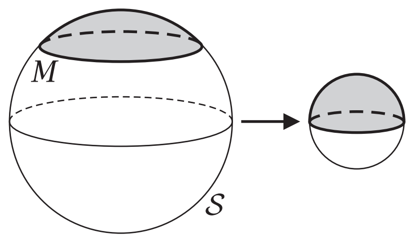

Figure 1. The Ricci flow (3.1) with $\lambda(t) = \lambda_0$ after the normalization.

By letting $\lambda(t)$ be a nontrivial function of t, we allow our results to include several cases which are, in a sense, more natural than the one just described. For instance, suppose we apply the Ricci flow to the sphere $\mathcal{S}$ in Figure 1. The manifold M will then evolve along with $\mathcal{S}$. This evolution will be described by equations (3.1) with $\lambda(t)$ equal to some nonconstant function $\lambda_1(t)$. We provide an illustration in Figure 2.

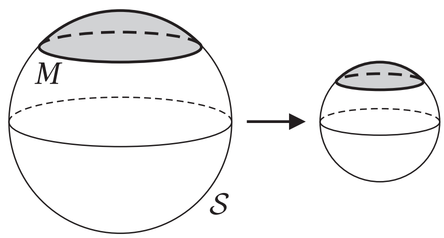

Figure 2. The Ricci flow (3.1) with $\lambda(t) = \lambda_1(t)$ before the normalization.

Let us normalize the Ricci flow on the sphere $\mathcal{S}$ so as to preserve the volume of $\mathcal{S}$. It is well-known that $\mathcal{S}$ will then remain unchanged for all t. Analogously, we can normalize the Ricci flow (3.1) with $\lambda(t) = \lambda_1(t)$ so as to preserve the volume of M. This will allow a better comparison with the situation shown in Figure 1. After such a normalization, the flow will keep M unchanged for all t.

3.2 Gradient estimates

Let us recollect some notation. The operator Δ is the Laplacian given by the metric $g(x,t)$. We write ∇ and $|\cdot|$ for the gradient and the norm with respect to $g(x,t)$. Our attention will be centered round the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad x \in M,\ t \in [0,T], \tag{3.2}$$

with the Neumann boundary condition

$$\frac{\partial}{\partial \nu} u(x,t) = 0, \qquad x \in \partial M,\ t \in [0,T]. \tag{3.3}$$

The results in this section still hold, with obvious modifications, if the solution $u(x,t)$ is only defined on $M \times (0,T]$. In this case, one just has to replace $u(x,t)$ and $g(x,t)$ with $u(x,t+\epsilon)$ and $g(x,t+\epsilon)$ for a sufficiently small $\epsilon > 0$, apply the corresponding theorem, and then let ϵ go to 0.

Our first result is a space-only estimate. It is analogous to (2.9).

Theorem 3.1. *Let* $\big(M, g(x,t)\big)_{t\in[0,T]}$ *be a solution to the Ricci flow* (3.1)*. Suppose* $u(x,t) : M \times [0,T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (3.2) *with the Neumann boundary condition* (3.3)*. Then the estimate*

$$\frac{|\nabla u|}{u} \leq \sqrt{\frac{1}{t} \log \frac{A}{u}}, \qquad x \in M,\ t \in (0,T], \tag{3.4}$$

holds with $A = \sup_M u(x,0)$.

Remark 3.2. Using the strong maximum principle and the Hopf maximum principle, one can show that A is actually equal to $\sup_{M\times[0,T]} u(x,t)$. Consequently, the right-hand side of (3.4) is well-defined.

We emphasize that the Laplacian Δ, the normal vector field $\frac{\partial}{\partial\nu}$, the gradient ∇, and the norm $|\cdot|$ appearing above depend on the parameter $t\in[0,T]$.

Proof. Introduce the function $P = t\frac{|\nabla u|^2}{u} - u\log\frac{A}{u}$. One may repeat the computation from the proof of Theorem 2.4 and conclude that

$$\left(\Delta - \frac{\partial}{\partial t}\right) P \geq 0 \tag{3.5}$$

for all $(x,t)\in M\times(0,T]$. Employing this inequality, we will demonstrate that P must be nonpositive. The assertion of the theorem will immediately follow.

Fix $\tau\in(0,T]$. Let us prove that the function P is nonpositive on $M\times[0,\tau]$. If P attains its largest value at the point $(x,0)$ for some $x\in M$, then P is less than or equal to $-u\log\frac{A}{u}$ computed at $(x,0)$. In this case, P must be nonpositive. Suppose this function attains its largest value at the point (x,t) for some x in the interior of M and some t in the interval $(0,\tau]$. We then use estimate (3.5) and the strong maximum principle. They imply P must also assume its largest value at $(x,0)$. As a consequence, P is nonpositive. Thus, we only have to consider the situation where this function has no maxima on $M\times[0,\tau]$ away from $\partial M\times(0,\tau]$. Unless this is the case, P cannot become strictly greater than 0 anywhere.

Let $(x_0,t_0)\in\partial M\times(0,\tau]$ be a point where the function P attains its largest value on $M\times[0,\tau]$. The Hopf maximum principle tells us that the inequality

$$\frac{\partial}{\partial\nu}P(x_0,t_0) > 0$$

holds true. But the Neumann boundary condition (3.3) and the second line of (3.1) imply

$$\begin{aligned}\frac{\partial}{\partial\nu}P &= t\left(\frac{\partial}{\partial\nu}|\nabla u|^2\right)\frac{1}{u} - t\frac{|\nabla u|^2}{u^2}\frac{\partial}{\partial\nu}u - \left(\frac{\partial}{\partial\nu}u\right)\log\frac{A}{u} + \frac{\partial}{\partial\nu}u \\ &= t\left(\frac{\partial}{\partial\nu}|\nabla u|^2\right)\frac{1}{u} = 2\frac{t}{u}\left(D_{\frac{\partial}{\partial\nu}}(\nabla u)\right)\nabla u = -2\frac{t}{u}\,\mathrm{II}(\nabla u,\nabla u) \\ &= -2\frac{t}{u}\,\lambda(t)|\nabla u|^2 \leq 0\end{aligned}$$

for all $(x,t)\in\partial M\times[0,\tau]$ (related computations appear in [25] and [27, Chapter IV]). Consequently, P must have a maximum on $M\times[0,\tau]$ away from $\partial M\times(0,\tau]$. We conclude that P is nonpositive on $M\times[0,\tau]$. Since the number $\tau\in(0,T]$ can be chosen arbitrarily, the same assertion holds on $M\times[0,T]$. The theorem follows at once. □

Remark 3.3. Consider the case where the metric $g(x,t)$ does not depend on t and equations (3.1) are not assumed. Suppose the Ricci curvature of M is nonnegative and ∂M is convex in the sense that the second fundamental form of ∂M is nonnegative definite. Then the solution $u(x,t)$ of problem (3.2)–(3.3) satisfies (3.4). This fact can be established by the same argument we used to prove the theorem. The computation leading to (3.5) in this case may be found in [15].

Our next estimate is similar to (2.16). Henceforth, the subscript t denotes the derivative in t. The number n is the dimension of the manifold M.

Theorem 3.4. *Let $\left(M,g(x,t)\right)_{t\in[0,T]}$ be a solution to the Ricci flow* (3.1)*. Consider a smooth positive function $u(x,t): M\times[0,T]\to\mathbb{R}$ satisfying the heat equation* (3.2) *with the Neumann boundary condition* (3.3)*. If $0\leq\mathrm{Ric}(x,t)\leq kg(x,t)$ for a fixed $k>0$ and all $(x,t)\in M\times[0,T]$, then the estimate*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq kn + \frac{n}{2t} \tag{3.6}$$

holds for all $(x,t)\in M\times(0,T]$.

Remark 3.5. We will make use of Lemma 2.6 in the arguments below. The proof of this lemma relies on local computations. Therefore, it prevails on manifolds with boundary.

Proof. Fix $\tau \in (0,T]$. Introduce the functions $f = \log u$ and $F_1 = t\left(|\nabla f|^2 - f_t\right)$. Let us pick a point $(x_0,t_0) \in M \times [0,\tau]$ where F_1 attains its maximum on $M \times [0,\tau]$. We will demonstrate that the inequality

$$F_1(x_0,t_0) \le t_0 kn + \frac{n}{2} \tag{3.7}$$

holds true. The assertion of the theorem will follow therefrom.

If $t_0 = 0$, then $F_1(x,t_0) = 0$ for every $x \in M$ and estimate (3.7) is evident. Consequently, we assume $t_0 > 0$. In accordance with Lemma 2.6 and our conditions on the Ricci curvature of M, the inequality

$$\left(\Delta - \frac{\partial}{\partial t}\right) F_1 \ge -2\nabla f \nabla F_1 + \frac{2a}{n}\frac{F_1^2}{t_0} - \frac{F_1}{t_0} - \frac{t_0 n}{2(1-a)}k^2$$

holds for all $a \in (0,1)$ at the point (x_0,t_0). Setting $a = \frac{1+kt_0}{1+2kt_0}$ like in the proof of Theorem 2.9 and using the quadratic formula, we see that

$$\left(\Delta - \frac{\partial}{\partial t}\right) F_1 + 2\nabla f \nabla F_1 \ge \left(F_1 + \frac{nkt_0(1+2kt_0)}{2(1+kt_0)}\right)\left(F_1 - t_0kn - \frac{n}{2}\right). \tag{3.8}$$

If (3.7) fails to hold, then the right-hand side of (3.8) must be strictly positive. We will now show this is impossible.

Suppose x_0 lies in the interior of M. The fact that (x_0,t_0) is a maximum point then yields $\Delta F_1(x_0,t_0) \le 0$, $\frac{\partial}{\partial t}F_1(x_0,t_0) \ge 0$, and $\nabla F_1(x_0,t_0) = 0$. Hence the right-hand side of (3.8) cannot be strictly positive. Suppose now x_0 lies in the boundary of M. If the right-hand side of (3.8) is indeed positive, then the Hopf maximum principle tells us that the inequality

$$\frac{\partial}{\partial \nu} F_1 > 0 \tag{3.9}$$

holds at (x_0,t_0). We will make a computation to show this cannot be the case.

Fix a system $\{y_1,\dots,y_n\}$ of local coordinates in a neighborhood U of the point x_0 demanding that $U \cap \partial M = \{x \in U \,|\, y_n(x) = 0\}$. We write g_{ij} and R_{ij} for the corresponding components of the metric and the Ricci tensor. Clearly, they depend on the parameter t. Without loss of generality, assume $\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{n-1}}$ are all orthogonal to $\frac{\partial}{\partial y_n}$ on the boundary with respect to $g(x,t_0)$. It is easy to see that

$$\frac{\partial}{\partial \nu} = -\sum_{i=1}^{n} \frac{g^{in}}{(g^{nn})^{\frac{1}{2}}}\frac{\partial}{\partial y_i} \tag{3.10}$$

in $U \cap \partial M$. Here, g^{ij} are the components of the matrix inverse to $(g_{ij})_{i,j=1}^n$.

The Neumann boundary condition (3.3) implies $\frac{\partial}{\partial \nu} f = 0$. Utilizing this fact, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \nu}F_1 &= t\left(\frac{\partial}{\partial \nu}|\nabla f|^2 - \frac{\partial}{\partial \nu}f_t\right) = t\left(2\left(D_{\frac{\partial}{\partial \nu}}(\nabla f)\right)\nabla f - \frac{\partial}{\partial \nu}f_t\right) \\
&= t\left(-2\,\mathrm{II}(\nabla f,\nabla f) + \left(\frac{\partial}{\partial t}\frac{\partial}{\partial \nu}\right)f - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \nu}f\right)\right) \\
&= t\left(-2\,\mathrm{II}(\nabla f,\nabla f) + \left(\frac{\partial}{\partial t}\frac{\partial}{\partial \nu}\right)f\right).
\end{aligned}$$

For related computations, see [25] and [27, Chapter IV]. According to (3.10) and the first formula in (3.1), the equality

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial \nu} &= -\frac{1}{g^{nn}}\sum_{i=1}^{n}\left(\left(\frac{\partial}{\partial t}g^{in}\right)(g^{nn})^{\frac{1}{2}} - \frac{\frac{\partial}{\partial t}g^{nn}}{2(g^{nn})^{\frac{1}{2}}}g^{in}\right)\frac{\partial}{\partial y_i} \\
&= -\sum_{i,j,l=1}^{n}\left(\frac{2R_{jl}g^{ji}g^{nl}}{(g^{nn})^{\frac{1}{2}}} - \frac{R_{jl}g^{jn}g^{nl}g^{in}}{g^{nn}(g^{nn})^{\frac{1}{2}}}\right)\frac{\partial}{\partial y_i}
\end{aligned}$$

holds in $U \cap \partial M$. A calculation based on the Codazzi equation and the second line in (3.1) then implies

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \nu} = R_{nn} g^{nn} \frac{\partial}{\partial \nu}$$

near x_0 at time t_0. Here, we make use of that fact that $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}}$ are orthogonal to $\frac{\partial}{\partial y_n}$ on the boundary with respect to $g(x, t_0)$. Combining the above equalities, we conclude that

$$\begin{aligned}\frac{\partial}{\partial \nu} F_1 &= t_0 \left(-2\,\mathrm{II}(\nabla f, \nabla f) + R_{nn} g^{nn} \frac{\partial}{\partial \nu} f \right) \\ &= -2t_0\,\mathrm{II}(\nabla f, \nabla f) = -2t_0 \lambda(t_0) |\nabla f|^2 \leq 0\end{aligned}$$

at the point (x_0, t_0). But this contradicts (3.9). Thus, the right-hand side of (3.8) cannot be strictly positive, and our assumption that (3.7) failed to hold must have been false.

Because (x_0, t_0) is a maximum point for F_1 on $M \times [0, \tau]$, it is easy to see that

$$F_1(x, \tau) \leq F_1(x_0, t_0) \leq t_0 kn + \frac{n}{2} \leq \tau kn + \frac{n}{2}$$

for any $x \in M$. Consequently,

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq kn + \frac{n}{2\tau}$$

at (x, τ). Since the number $\tau \in (0, T]$ can be chosen arbitrarily, this yields the assertion of the theorem. $\square$

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